NEWMAN'S THEOREM IN THE RIEMANNIAN CATEGORY

BY

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ABSTRACT. In 1931 M. H. A. Newman proved that if M is a connected topological manifold with metric d, there exists a number $\epsilon > 0$, depending only upon M and d, such that every compact Lie group G acting effectively on M has at least one orbit of diameter at least ϵ. Aside from isolated results nothing appears to be known about ϵ. In order to learn more about the invariant ϵ, attention is restricted here to groups of isometries on a Riemannian manifold. It is found that the invariant ϵ of M is connected with the notion of convexity introduced by J. H. C. Whitehead in 1932.

1. **Introduction.** In 1931 M. H. A. Newman proved the following result.

THEOREM (NEWMAN) [7], [2]). *If M is a connected topological manifold with metric d, there exists a number $\epsilon > 0$, depending only upon M and d, such that every compact Lie group G acting effectively on M has at least one orbit of diameter at least ϵ.*

Later P. A. Smith [8], [1, III.9] gave a generalization of Newman's theorem to cohomology manifolds with fixed coverings. The several proofs of Newman's theorem do not shed much light on the nature of the invariant ϵ. If M is the unit sphere S^{n-1} with the standard metric in R^n and if we restrict our attention to *orthogonal* effective actions of G on S^{n-1}, then it can be shown that $\epsilon = 2\pi/3$ is best possible. However, aside from isolated results like this, nothing appears to be known about ϵ.

In order to learn more about the invariant ϵ, a good starting place appears to be the Riemannian category; in other words we restrict our attention to groups of isometries on a Riemannian manifold M. We have found that here the invariant ϵ of M is connected with the notion of convexity introduced by Whitehead in [9]. For each $p \in M$ we introduce a positive number r_p called the *radius of convexity of M at p* (compare with [3]). If K is a compact subset of M, $\inf_K r_p > 0$. Our results are easily stated for the case when M is compact.

THEOREM (THEOREMS 2 AND 4). *Suppose M is a compact Riemannian manifold with $r(M) = \inf_{p \in M} r_p$. If G is a compact Lie group of isometries acting effectively on M, there exists at least one orbit of diameter at least $r(M)$. In fact, if G acts without fixed points, every orbit has diameter at least $r(M)$.*

If M is not compact, our results are incomplete for the fixed point case as we cannot deal in an entirely satisfactory way with the situation where the isometry group G is a finite group of odd order (Theorem 3).

The techniques of this paper were suggested to us by [6]. The authors are grateful to Professor Hans Samelson for his valuable suggestions and to their colleagues H. T. Ku and David Hoffman for the proof of Theorem 4.

2. Preliminaries and the fixed point free case. Let M be a Riemannian manifold with metric $d(-, -)$.

DEFINITION. A subset S of M will be called *very strongly convex* if the following hold:

(1) For all $x, y \in S$, there exists a unique geodesic g_{xy} in S joining x to y, and length $g_{xy} = d(x, y)$.

(2) Moreover g_{xy} is the only geodesic in M joining x to y having length equal to $d(x, y)$.

(3) If g_{uv} is a geodesic in S joining u to v and q is any point of S, $d(q, z) <$ Max$\{d(q, u), d(q, v)\}$ for each interior point z of g_{uv}.

Our notion of very strongly convex was suggested by the *stark konvex* of [3, p. 159]. Recall that S is called *stark konvex* if it and all open balls contained in it satisfy condition (1). It is easy to verify that very strongly convex implies stark konvex. We shall say more about this relationship later.

PROPOSITION 1. *Let S be a compact very strongly convex subset of M. If G is a compact Lie group of isometries acting effectively on M and with no fixed points in S, then S contains no orbit of the action.*

PROOF. If S contains an orbit, it must contain some orbit, say $G(a)$, of minimum diameter. Since $G(a)$ is not a fixed point, there exists $c \in G(a)$, $c \neq a$. Let g_{ac} be the unique geodesic in S joining a to c with length $g_{ac} = d(a, c)$. Let b be any interior point of g_{ac}. We show that

(i) $G(b) \subset S$, and

(ii) diam $G(b) <$ diam $G(a)$,

which is, of course, a contradiction.

(i) Let $\lambda \in G$. Then λg_{ac} is a geodesic joining λa to λc where $\lambda a, \lambda c \in G(a) \subset S$. Now

$$\text{length } \lambda g_{ac} = \text{length } g_{ac} = d(a, c) = d(\lambda a, \lambda c).$$

So by condition (2) of very strong convexity, λg_{ac} must be the unique geodesic in S joining λa to λc. Consequently $\lambda b \in \lambda g_{ac} \subset S$ and $G(b) \subset S$ since λ was an arbitrary element of G.

(ii) Choose $\mu \in G$ such that $d(\mu b, b) = \text{diam } G(b)$. Consider μb and the geodesic g_{ac} in S. By condition (3) of very strong convexity,

(A) $d(\mu b, b) < \text{Max}\{d(\mu b, a), d(\mu b, c)\}$. Considering c and μg_{ac} in S,

(B) $d(c, \mu b) < \text{Max}\{d(c, \mu a), d(c, \mu c)\} \leqslant \text{diam } G(a)$ and considering a and μg_{ac} in S,

(C) $d(a, \mu b) < \text{Max }\{d(a, \mu a), d(a, \mu c)\} \leqslant \text{diam } G(a)$. So by (A), (B), and (C),

$$\text{diam } G(b) = d(\mu b, b) < \text{diam } G(a).$$

For $p \in M$ and $\rho > 0$, let

$$B(p, \rho) = \{x \in M/d(x, p) < \rho\}$$

denote the open ball of radius ρ about p.

DEFINITION. *The radius of convexity of M at a point p is denoted by* r_p and defined as follows:

$$r_p = \sup\{\rho/B(p, \rho) \text{ is very strongly convex and } \overline{B(p, \rho)} \text{ is compact}\}.$$

REMARK. Alternatively

$$r_p = \sup\{\rho/B(p, \rho) \text{ is very strongly convex and compact}\}.$$

In general, our radius of convexity is smaller than the *Konvexitätsradius* of [3, p. 162]. The *Konvexitätsradius* of M at p is denoted by r_p^* and defined as follows:

$$r_p^* = \sup\{\rho/B(p, \rho) \text{ is stark konvex}\}.$$

If S^{n-1} is the unit sphere with the standard metric in R^n,

$$r_p^* = r_p = \pi/2 \quad \text{for } n = 2,$$

$$r_p^* = \pi/2, r_p = \pi/4 \quad \text{for } n > 2.$$

We have the following comparison between r_p and r_p^*.

PROPOSITION 2. $r_p^*/3 \leqslant r_p \leqslant r_p^*$ *where* r_p^* *is the Gromoll-Klingenberg-Meyer "Konvexitätsradius" of M at p.*

PROOF. Since a very strongly convex ball is stark konvex, the right-hand inequality is immediate.

Let $0 < \rho < r_p^*$. It is sufficient to show that $B_0 = B(p, \rho/3)$ is very strongly

convex. Clearly B_0 satisfies condition (2) of very strong convexity and $\overline{B}_0$ is compact. So suppose g_{uv} is a geodesic in B_0 from u to v, and q a point of B_0 such that there exists an interior point z of g_{uv} with

$$d(q, z) = \max_{w \in g_{uv}} d(q, w).$$

Let $h(t)$ be the unique geodesic in B_0 from z to q and choose q' on $h(t)$ slightly past q. Now

$$d(q', z) = d(q'\, q) + d(q, z) \geqslant d(q', q) + d(q, u)$$

and $d(q'\, q) + d(q, u) > d(q', u)$ since, otherwise, the unique geodesic from q' to u would have to coincide with $-h(t)$ from q' to q and, consequently, from q' to z which is clearly absurd. So $d(q', z) > d(q', u)$ and similarly, $d(q', z) > d(q', v)$. Finally choose z' on g_{uv} such that

$$d(q', z') = \max_{w \in g_{uv}} d(q', w).$$

Now g_{uv} does not lie in the open ball around q' with radius $d(q', z')$. However, this is a contradiction since this ball and any slightly larger ball (which necessarily contains g_{uv}) are stark konvex.

COROLLARY. (1) $r_p > 0$ for all $p \in M$.
(2) If $r_{p_0} = +\infty$ for some $p_0 \in M$, then $r_p = +\infty$ for all $p \in M$.
(3) r_p is a continuous function of p in M.

PROOF. (1) and (2) follow immediately since r_p^* has these properties [3]. The continuity of r_p may be shown in exactly the same way as the continuity of r_p^* is shown in [3, p. 162].

THEOREM 1. *Let M be a Riemannian manifold. Then if G is a compact Lie group of isometries acting effectively and with no fixed points on M,* diam $G(p) \geqslant r_p$, *each $p \in M$. In particular if there exists $p \in M$ with $r_p = +\infty$, every compact Lie group of isometries acting on M must have a fixed point.*

PROOF. Let γ be any number such that $0 < \gamma < r_p$. Then $\overline{B(p, \gamma)}$ is a compact very strongly convex subset of M. By Proposition 1, diam $G(p) > \gamma$.

REMARK. The last part of Theorem 1 implies a result of E. Cartan [4, p. 75].

THEOREM 2. *Let M be a Riemannian manifold. Define*

$$r(M) = \inf_{p \in M} r_p, \quad R(M) = \sup_{p \in M} r_p.$$

Then if G is a compact Lie group of isometries acting effectively and with no fixed points on M, we have the following:

(1) *Every orbit has diameter at least $r(M)$.*

(2) *If $\widetilde{R}$ is any number such that $0 < \widetilde{R} < R(M)$, there exists at least one orbit of diameter at least $\widetilde{R}$.*

In particular if $r(M) = +\infty$, every compact Lie group of isometries acting on M must have a fixed point.

PROOF. Theorem 2 is, of course, an immediate consequence of Theorem 1.

EXAMPLE. If M is the unit sphere S^{n-1} with the standard metric, $r(M) = R(M) = \pi/4$ $(n \geqslant 3)$. However, as we noted in the Introduction, $\epsilon = 2\pi/3$ is best possible in this case. In this case we obtain a better bound from Theorem 5.

We call a Riemannian manifold M *homogeneous* if the group of isometries of M acts transitively on M. If M is homogeneous, it is easy to verify that r_p is a fixed positive constant (possibly $+\infty$) for all $p \in M$. Consequently,

REMARK. If M is either compact or homogeneous, then $r(M) > 0$.

3. Fixed points.

THEOREM 3. *Suppose M is a Riemannian manifold. Let $\widetilde{R}$ be any number, $0 < \widetilde{R} < R(M)$. Suppose G is a compact Lie group of isometries acting effectively (and possibly with fixed points) on M. Then if either*

(1) *G is not a finite group of odd order, or*

(2) *M has nonpositive sectional curvatures,*

there exists at least one orbit of diameter larger than $\widetilde{R}/2$.

PROOF. Choose $p \in M$ such that $r_p > \widetilde{R}$. Then $\overline{B(p, \widetilde{R}/2)}$ is a compact very strongly convex subset of M. If $\overline{B(p, \widetilde{R}/2)}$ contains no fixed points of the action, it follows directly from Proposition 1 that diam $G(p) > \widetilde{R}/2$.

Suppose then $f \in \overline{B(p, \widetilde{R}/2)}$ with $G(f) = f$. Then $\overline{B(f, \widetilde{R}/2)}$ is stark konvex. Since f is a fixed point of G, G acts as a group of differentials and, in fact, orthogonally and effectively on the tangent space M_f. Let

$$S^* = S(0, \widetilde{R}/2) = \{x^* \in M_f / d^*(x^*, 0) = \widetilde{R}/2\}$$

be the sphere of radius $\widetilde{R}/2$ about 0 in M_f, where $d^*(-, -)$ is the euclidean metric on M_f. Since $\overline{B(p, \widetilde{R}/2)}$ is stark konvex, the exponential map, exp, is a diffeomorphism of $S^* = S(0, \widetilde{R}/2)$ onto

$$S = S(f, \widetilde{R}/2) = \{x \in M/d(x, f) = \widetilde{R}/2\}.$$

Moreover exp is equivariant with respect to the actions of G on S^* and S.

Case (1) *G is not a finite group of odd order.* We restrict our attention to a subgroup H of G with $H \cong Z_2$. The action of G, and hence H, is orthogonal and effective on S^* so there exists an orbit of H consisting of $\{x_1^*, -x_1^*\}$, i.e.,

a pair of antipodal points of S^*. Now $d^*(x_1^*, -x_1^*) = \widetilde{R}$. Let $x_1 = \exp x_1^*, y_1 = \exp(-x_1^*)$. So $x_1, y_1 \in S$. Since exp is equivariant, $\{x_1, y_1\}$ is an orbit of H on S. Moreover $d(x_1, y_1) = \widetilde{R}$, since the unique geodesic joining x_1 to y_1 is the image under exp of the straight line segment through x_1^*, 0 and $-x_1^*$ in M_f. Hence

$$\text{diam } G(x_1) \geqslant \text{diam } H(x_1) = \widetilde{R} > \widetilde{R}/2.$$

Case (2) *M has nonpositive sectional curvatures.* We may assume G is finite of odd order and restrict our attention to a subgroup H of G with $H \cong Z_p$, p an odd prime. Since H acts orthogonally on S^*, H acts invariantly and effectively on some great circle C^* of S^*. It is easy to verify that an orbit of H on C^* has diameter

$$\widetilde{R} \sin\left[\left(\frac{p-1}{p}\right)\pi/2\right] \geqslant \widetilde{R} \sin\left[\frac{2}{3}\frac{\pi}{2}\right] = \frac{\sqrt{3}}{2}\widetilde{R}.$$

So let $x_1^*, y_1^* \in C^*$ be on $H(x_1^*)$ with

$$d^*(x_1^*, y_1^*) = \text{diam } H(x_1^*) \geqslant \frac{\sqrt{3}}{2}\widetilde{R}.$$

Then $x_1 = \exp x_1^*, y_1 = \exp y_1^*$ are on $H(x_1) \subset S \subset M$. Since M has nonpositive sectional curvatures, it follows [4, p. 73] that

$$d(\exp x_1^*, \exp y_1^*) \geqslant d^*(x_1^*, y_1^*).$$

So

$$\text{diam } G(x_1) \geqslant \text{diam } H(x_1) \geqslant d(x_1, y_1) \geqslant d^*(x_1^*, y_1^*) \geqslant \frac{\sqrt{3}}{2}\widetilde{R} > \widetilde{R}/2.$$

REMARK. It is likely that Theorem 3 can be extended to the case where the sectional curvatures of M are bounded. In any event, if M is compact we have the following complete result.

THEOREM 4. *Suppose M is a compact Riemannian manifold. Then if G is a compact Lie group of isometries acting effectively (and possibly with fixed points) on M, there exists at least one orbit of diameter at least $r(M)$.*

PROOF. If the fixed point set F is empty, the result follows immediately from Theorem 2. Choose $p \in M$ such that

$$d(p, F) = \max_{x \in M} d(x, F) = l.$$

We shall show that $r_p \leqslant l$ which, using Proposition 1, implies that diam $G(p) \geqslant r_p \geqslant r(M)$. We suppose then that $r_p > l$ and reach a contradiction. There exists $f \in F$ with $d(p, f) = d(p, F)$. Now f must be unique for suppose there exists $f' \in F, f' \neq f$ with $d(p, f') = d(p, f) = l < r_p$. Let $l < \rho < r_p$ and $B_0 = B(p, \rho)$.

Then B_0 is very strongly convex. Let $g_{ff'}$ be the unique geodesic in B_0 joining f to f'. As F is totally geodesic, $g_{ff'} \subset F$. Therefore for any interior point u of $g_{ff'}$,

$$d(p, u) \geqslant \min_{v \in F} d(p, v) = d(p, f) = d(p, f').$$

But this contradicts the fact that B_0 is very strongly convex. Hence f is unique.

Let $h(t)$ be the unique geodesic in B_0 from f to p, $h(t_0) = p$. Take a sequence p_n on $h(t)$ past p and converging to p. Let f_n be a point of F with $d(p_n, f_n) = d(p_n, F)$ and note that $d(p_n, f_n) \leqslant d(p, f)$. Since f is the unique point of F with $d(p, F) = d(p, f)$, a subsequence of the f_n's converge to f. Consider $G(t, z) = d(h(t), z)$ in a small neighborhood of (t_0, f). Since we are within B_0 and G stays away from zero, G is a C^∞ function of t and z [5, p. 166]. Clearly $\partial G/\partial t > 0$ at (t_0, f) and, consequently, in a small enough neighborhood of (t_0, f). It follows that for some n, $d(p_n, f_n) > d(p, f_n) > d(p, f)$. So we have a contradiction and therefore it must be that $r_p \leqslant l$.

We used our notion of radius of convexity rather than that of [3] since it simplified our proofs and gave us a very concise statement of Theorem 1. It is of course possible to state all our results in terms of the Gromoll-Klingenberg-Meyer "Konvexitätsradius" using the left-hand inequality of Proposition 2. However, in some cases, sharper results may be obtained by direct arguments. For example we can show the following.

THEOREM 5. *Suppose M is a compact Riemannian manifold with* $r^*(M) = \inf_{p \in M} r_p^*$. *If G is a compact Lie group of isometries acting effectively on M, there exists at least one orbit of diameter at least* $r^*(M)$. *In fact, if G acts without fixed points, every orbit has diameter at least* $2r^*(M)/3$.

ADDED IN PROOF. Karsten Grove and Hermann Karcher [Math. Z. 132 (1973)] have investigated C^1-close group actions on compact Riemannian manifolds and have obtained in this setting several Newman type results.

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